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OSCILLATIONS OF THE SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS OF NEUTRAL TYPE

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Abstract

In this paper nonlinear hyperbolic equations of neutral type of the form

$$\frac{\partial^2}{\partial t^2}[u(x,t) + \lambda(t)u(x,t-\tau)] - [\Delta u(x,t) + \mu(t)\Delta u(x,t-\sigma)] + c(x,t,u) = f(x,t), \qquad (x,t) \in \Omega \times (0,\infty) \equiv G,$$

are considered, where τ, $\sigma = \text{const} > 0$, with boundary conditions

$$\frac{\partial u}{\partial n} + \gamma(x,t)u = g(x,t), \qquad (x,t) \in \partial\Omega \times [0,\infty)$$

or

$$u = 0, \qquad (x,t) \in \partial\Omega \times [0,\infty).$$

Under certain constraints on the coefficients of the equation and the boundary conditions, sufficient conditions for oscillation of the solutions of the problems considered are obtained.

1. Introduction

In the last few years results related to the oscillatory properties and asymptotic behaviour of the solutions of some classes of hyperbolic equations were published. We shall mention especially the work of K. Kreith, T. Kusano and N. Yoshida [4] in which sufficient conditions for oscillation of the solutions of the nonlinear hyperbolic equation

$$u_{tt} - \Delta u + c(x,t,u) = f(x,t)$$

considered in a cylindrical domain are obtained. Oscillatory properties of the solutions of hyperbolic differential equations with a deviating argument were investigated in the works of D. Georgiou, K. Kreith [2], D. Georgiou [3]. Hyperbolic differential equations with *maxima* were investigated in the work of D. Mishev [6] and some conditions for oscillation of the solutions of hyperbolic equations of neutral type were obtained by D. Mishev and D. Bainov in [7], [8].

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2. Preliminary notes

In the present paper sufficient conditions for oscillation of the solutions of nonlinear hyperbolic equations of neutral type of the form

$$(1)\quad \frac{\partial^2}{\partial t^2}[u(x,t)+\lambda(t)u(x,t-\tau)]-[\Delta u(x,t)+\mu(t)\Delta u(x,t-\sigma)]+ \\ +c(x,t,u)=f(x,t), \qquad (x,t)\in\Omega\times(0,\infty)\equiv G,$$

are obtained, where τ, $\sigma=\text{const}>0$, $\Delta u(x,t)=\sum\limits_{i=1}^{n}u_{x_ix_i}(x,t)$ and Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary.

Consider boundary conditions of the form

$$(2)\qquad \frac{\partial u}{\partial n}+\gamma(x,t)u=g(x,t), \qquad (x,t)\in\partial\Omega\times[0,\infty)$$

$$(3)\qquad u=0, \qquad (x,t)\in\partial\Omega\times[0,\infty)$$

We shall say that conditions (H) are satisfied if the following conditions hold:

H1. $\lambda(t)\in C^2([0,\infty);[0,\infty))$,
$\mu(t)\in C([0,\infty);\mathbb{R})$,
H2. $c(x,t,u)\in C(G\times\mathbb{R};\mathbb{R})$,
H3. $c(x,t,-u)=-c(x,t,u), \qquad (x,t,u)\in G\times(0,\infty)$,
H4. $c(x,t,u)\geq p(t)\cdot h(u), \qquad (x,t,u)\in G\times(0,\infty)$,

where $p(t)$ is a continuous and positive function in the interval $(0,\infty)$ and $h(u)$ is a continuous, positive and convex function in the same interval $(0,\infty)$.

H5. $f(x,t)\in C(G;\mathbb{R})$
H6. $g(x,t)\in C(\partial\Omega\times[0,\infty);\mathbb{R})$
H7. $\gamma(x,t)\in C(\partial\Omega\times[0,\infty);[0,\infty))$.

Definition 1. The solution $u(x,t)\in C^2(G)\cap C^1(\bar{G})$ of problem (1), (2) ((1, (3)) is said to oscillate in the domain G if for any positive number μ there exists a point $(x_0,t_0)\in\Omega\times[\mu,\infty)$, such that the equality $u(x_0,t_0)=0$ holds.

In the subsequent theorems sufficient conditions for oscillation of the solutions of problems (1), (2) and (1), (3) in the domain G are obtained. We shall note that in the work of K. Kreith, T. Kusano, N. Yoshida [4] conditions are obtained for the oscillation of the solutions only of problem (1), (2) in the case when $\lambda(t)\equiv 0$, $\mu(t)\equiv 0$ and $\gamma(x,t)\equiv 0$.

Introduce the following notation:

$$F(t) = \frac{1}{|\Omega|} \cdot \int_\Omega f(x,t)\,dx, \qquad t > 0, \tag{4}$$

$$G(t) = \frac{1}{|\Omega|} \cdot \int_{\partial\Omega} g(x,t)\,ds, \qquad t > 0, \tag{5}$$

where $|\Omega| = \int_\Omega dx$.

With any solution $u(x,t) \in C^2(G) \cap C^1(\bar{G})$ of problem (1), (2) we associate the function

$$v(t) = \frac{1}{|\Omega|} \cdot \int_\Omega u(x,t)\,dx, \qquad t > 0. \tag{5}$$

Lemma 1. *Let conditions (H) hold and let $u(x,t)$ be a positive solution of problem (1), (2) in the domain G. Then the function $v(t)$ defined by (5) satisfies the differential inequality of neutral type*

$$\frac{d^2}{dt^2}[v(t) + \lambda(t)v(t-\tau)] + p(t)h(v(t)) \leqq G(t) + \mu(t)G(t-\sigma) + F(t), \tag{6}$$

$t \geqq t_0$, where t_0 is a sufficiently large positive number.

Proof: Let $u(x,t)$ be a positive solution in the domain G of problem (1), (2) and $t_0 = \max\{\tau, \sigma\}$. Then $u(x, t-\tau) > 0$ and $u(x,t-\sigma) > 0$ for $(x,t) \in \Omega \times [t_0, \infty)$. We integrate both sides of equation (1) with respect to x over the domain Ω and obtain for $t \geqq t_0$:

$$\begin{aligned} \frac{d^2}{dt^2}\left[\int_\Omega u(x,t)\,dx + \lambda(t)\int_\Omega u(x,t-\tau)\,dx\right] - \left[\int_\Omega \Delta u(x,t)\,dx + \right. \\ \left. + \mu(t)\int_\Omega \Delta u(x,t-\sigma)\,dx\right] + \int_\Omega c(x,t,u)\,dx = \int_\Omega f(x,t)\,dx. \end{aligned} \tag{7}$$

From Green's formula and condition H7 it follows that

$$\int_\Omega \Delta u(x,t)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,ds = \int_{\partial\Omega}[g(x,t) - \gamma(x,t)u]\,ds \leq \int_{\partial\Omega} g(x,t)\,ds \tag{8}$$

$$\begin{aligned} \int_\Omega \Delta u(x,t-\sigma)\,dx &= \int_{\partial\Omega}\frac{\partial u}{\partial n}(x,t-\sigma)\,ds = \\ &= \int_{\partial\Omega}[g(x,t-\sigma) - \gamma(x,t-\sigma)\cdot u(x,t-\sigma)]\,ds \leqq \int_{\partial\Omega} g(x,t-\sigma)\,ds \end{aligned} \tag{9}$$

Moreover, from condition H4 and Jensen's inequality it follows that

$$\begin{aligned}(10)\quad &\int_\Omega c(x,t,u)\,dx \geqq p(t)\int_\Omega h(u(x,t))\,dx \geqq \\ &\geqq p(t)h\left(\int_\Omega u(x,t)\,dx\cdot\left(\int_\Omega dx\right)^{-1}\right)\cdot\int_\Omega dx = p(t)\cdot h(v(t))\cdot|\Omega|\end{aligned}$$

Using (8)-(10) and condition H1, from (7) we obtain

$$\frac{d^2}{dt^2}[v(t)+\lambda(t)v(t-\tau)] \leqq G(t)+\mu(t)G(t-\sigma)+F(t)-p(t)\cdot h(v(t)),$$

which proves Lemma 1. ■

3. Main results

Theorem 1. *Let conditions (H) hold and let the differential inequalities of neutral type*

$$\frac{d^2}{dt^2}[v(t)+\lambda(t)v(t-\tau)]+p(t)\cdot h(v(t)) \leqq G(t)+\mu(t)G(t-\sigma)+F(t) \tag{11}$$

$$\frac{d^2}{dt^2}[v(t)+\lambda(t)v(t-\tau)]+p(t)\cdot h(v(t)) \leqq G(t)-\mu(t)\cdot G(t-\sigma)-F(t) \tag{12}$$

have no eventually positive solutions. Then each solution $u(x,t)$ of problem (1), (2) oscillates in the domain G.

Proof: Let $\mu > 0$ be a positive number. Suppose that the assertion of the theorem is not true and let $u(x,t)$ be a solution of problem (1), (2) without zeroes in the domain $G_\mu = \Omega \times [\mu, \infty)$. If $u(x,t) > 0$ for $(x,t) \in G_\mu$, then from Lemma 1 it follows that the function $v(t)$ defined by (5) is a positive solution of inequality (11) for $t \geqq t_0 + \mu$, i.e it is an eventually positive solution of (11) which contradicts the assumption of the theorem. If $u(x,t) < 0$ for $(x,t) \in G_\mu$, then the function $-u(x,t)$ is a positive solution of the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[u+\lambda(t)u(x,t-\tau)] - [\Delta u + \mu(t)\Delta u(x,t-\sigma)] + \\ \qquad\qquad\qquad + c(x,t,u) = -f(x,t),\ (x,t)\in G \\ \dfrac{\partial u}{\partial n} + \gamma(x,t)u = -g(x,t), \qquad (x,t)\in\partial\Omega\times[0,\infty). \end{cases}$$

From Lemma 1 it follows that the function $\frac{1}{|\Omega|}\int_\Omega(-u(x,t))\,dx$ is a positive solution of inequality (12) for $t \geqq t_0 + \mu$ which also contradicts the assumption of the theorem. Thus Theorem 1 is proved. ■

Now we shall investigatge the oscillatory properties of the solutions of problem (1), (3). Consider in the domain Ω the following Dirichlet problem:

$$\begin{cases} \Delta U + \alpha U = 0 \text{ in } \Omega \\ U|_{\partial\Omega} = 0 \end{cases}$$

where $\alpha = $ const. It is well known [**1**] that the smallest eigenvalue α_0 is positive and the corresponding eigenfunction $\varphi(x)$ can be chosen to satisfy the inequality $\varphi(x) > 0$ for $x \in \Omega$.

With any solution $u(x,t) \in C^2(G) \cap C^1(\bar{G})$ of problem (1), (3) we associate the function

$$w(t) = \int_\Omega u(x,t)\varphi(x)\,dx \cdot \left(\int_\Omega \varphi(x)\,dx\right)^{-1}, \qquad t > 0 \tag{13}$$

We shall note that a similar averaging was first used by N. Yoshida in the work [**10**].

Lemma 2. *Let conditions H1-H6 hold and let $u(x,t)$ be a positive solution in the domain G of problem (1), (3). Then the function $w(t)$ defined by (13) satisfies the differential inequality of neutral type*

$$\begin{aligned} &\frac{d^2}{dt^2}[w(t) + \lambda(t)w(t-\tau)] + \alpha_0 w(t) + \alpha_0\mu(t)\cdot w(t-\sigma) + \\ &\quad + p(t)\cdot h(w(t)) \leqq \int_\Omega f(x,t)\varphi(x)\,dx \cdot \left(\int_\Omega \varphi(x)\,dx\right)^{-1}, \qquad t \geqq t_0, \end{aligned} \tag{14}$$

where t_0 is a sufficiently large positive number.

Proof: Let $u(x,t)$ be a positive solution in the domain G of problem (1), (3) and $t_0 = \max\{\tau,\sigma\}$. Then $u(x,t-\tau) > 0$ and $u(x,t-\sigma) > 0$ for $(x,t) \in \Omega \times (t_0,\infty)$. Multiply both sides of equation (1) by the eigenfunction $\varphi(x)$ of the Dirichlet problem and integrate with respect to x over the domain Ω. For $t \geqq t_0$ we obtain

$$\begin{aligned} &\frac{d^2}{dt^2}\left[\int_\Omega u(x,t)\varphi(x)\,dx + \lambda(t)\int_\Omega u(x,t-\tau)\varphi(x)\,dx\right] - \\ &\quad - \left[\int_\Omega \Delta u(x,t)\varphi(x)\,dx + \mu(t)\int_\Omega \Delta u(x,t-\sigma)\varphi(x)\,dx\right] + \\ &\qquad + \int_\Omega c(x,t,u)\varphi(x)\,dx = \int_\Omega f(x,t)\varphi(x)\,dx. \end{aligned} \tag{15}$$

From Green's formula it follows that

$$(16)\quad \int_\Omega \Delta u(x,t)\varphi(x)\,dx = \int_\Omega u(x,t)\Delta\varphi(x)\,dx = \\ = -\alpha_0 \cdot \int_\Omega u(x,t)\varphi(x)\,dx = -\alpha_0 w(t)\cdot \int_\Omega \varphi(x)\,dx$$

$$(17)\quad \int_\Omega \Delta u(x,t-\sigma)\varphi(x)\,dx = \int_\Omega u(x,t-\sigma)\Delta\varphi(x)\,dx = \\ = -\alpha_0 \int_\Omega u(x,t-\sigma)\varphi(x)\,dx = -\alpha_0 w(t-\sigma)\cdot \int_\Omega \varphi(x)\,dx,$$

where α_0 is the smallest eigenvalue. Moreover, from condition H4 and Jensen's inequality it follows that

$$(18)\quad \int_\Omega c(x,t,u)\varphi(x)\,dx \geqq p(t)\int_\Omega h(u)\varphi(x)\,dx \geqq \\ \geqq p(t)\cdot h\left(\int_\Omega u(x,t)\varphi(x)\,dx \cdot \left(\int_\Omega \varphi(x)\,dx\right)^{-1}\right)\cdot \int_\Omega \varphi(x)\,dx = \\ = p(t)\cdot h(w(t))\cdot \int_\Omega \varphi(x)\,dx$$

Using (16)-(18) and condition H1, from (15) we obtain

$$\frac{d^2}{dt^2}[w(t)+\lambda(t)w(t-\tau)] \leqq -\alpha_0[w(t)+\mu(t)w(t-\sigma)]- \\ - p(t)\cdot h(w(t)) + \int_\Omega f(x,t)\varphi(x)\,dx\cdot\left(\int_\Omega \varphi(x)\,dx\right)^{-1},$$

which completes the proof of Lemma 2. ■

Introduce the notation

$$(19)\qquad F_1(t) = \int_\Omega f(x,t)\varphi(x)\,dx\cdot\left(\int_\Omega \varphi(x)\,dx\right)^{-1}, \qquad t>0$$

Analogously to Theorem 1 the following theorem is proved.

Theorem 2. *Let conditions H1-H5 hold and let the differential inequalities of neutral type*

$$(20)\quad \frac{d^2}{dt^2}[w(t)+\lambda(t)w(t-\tau)] + \alpha_0[w(t)+\mu(t)w(t-\sigma)]+ \\ + p(t)\cdot h(w(t)) \leqq F_1(t), \qquad t\geqq t_0,$$

$$(21)\quad \frac{d^2}{dt^2}[w(t)+\lambda(t)w(t-\tau)]+\alpha_0[w(t)+\mu(t)w(t-\sigma)]+ \\ +p(t)\cdot h(w(t)) \leqq -F_1(t), \qquad t \geqq t_0$$

have no eventually positive solutions. Then each solution $u(x,t)$ of problem (1), (3) oscillates in the domain G.

From the theorems proved above it follows that the finding of sufficient conditions for oscillation of the solutions of equation (1) in the domain G is reduced to the investigation of the oscillatory properties of differential inequalities of neutral type of the form

$$(22)\quad \frac{d^2}{dt^2}[x(t)+\lambda(t)x(t-\tau)]+q_0(t)x(t)+q(t)x(t-\sigma)+ \\ +p(t)\cdot h(x(t)) \leqq H(t), \qquad t \geqq t_0$$

We shall say that condition (A) are satisfied if the following conditions hold:

A1. $\lambda(t) \in C^2([t_0,\infty);\ [0,\infty))$,
A2. $q_0(t),\ q(t) \in C([t_0,\infty);\ [0,\infty))$,
A3. $p(t) \in C([t_0,\infty);\ [0,\infty))$,
A4. $h(u) \in C(\mathbb{R};\ \mathbb{R})$, $h(u) > 0$ for $u > 0$,
A5. $H(t) \in C([t_0,\infty);\ \mathbb{R})$.

Theorem 3. *Let conditions (A) hold as well as the condition*

$$(23)\qquad \liminf_{t\longrightarrow\infty}\frac{1}{t-t_1}\int_{t_1}^{t}(t-s)\cdot H(s)\,ds=-\infty$$

for $t_1 \geqq t_0$. Then the differential inequality (22) has no eventually positive solutions.

Proof: Suppose that this is not true and let $x(t)$ be a positive solution of inequality (22) defined in the interval $[t_1,\infty)$, where $t_1 \geqq t_0$. Then in virtue of conditions A2-A4 we obtain for $t \geqq t_2$ ($t_2 \geqq t_1+\max\{\sigma,\tau\}$)

$$\frac{d^2}{dt^2}[x(t)+\lambda(t)x(t-\tau)] \leqq H(t)-q_0(t)x(t)-q(t)x(t-\sigma)- \\ -p(t)\cdot h(x(t)) \leqq H(t).$$

We integrate twice the above inequality over the interval $[t_2,t]$, $t > t_2$ and obtain

$$x(t)+\lambda(t)x(t-\tau) \leqq C_1+C_2(t-t_2)+\int_{t_2}^{t}\left[\int_{t_2}^{\rho}H(s)\,ds\right]d\rho,$$

where C_1, $C_2 =$ const. Since

$$\int_{t_2}^{t}\left[\int_{t_2}^{\rho} H(s)\,ds\right]d\rho = \int_{t_2}^{t}(t-s)H(s)\,ds,$$

dividing both sides of last inequality by $t - t_2 > 0$, we obtain

$$\frac{x(t)+\lambda(t)x(t-\tau)}{t-t_2} \leqq \frac{C_1}{t-t_2} + C_2 + \frac{1}{t-t_2}\int_{t_2}^{t}(t-s)H(s)\,ds \tag{24}$$

Then for $t \to \infty$ from (24), making use of condition (23), we obtain that

$$\liminf_{t\longrightarrow\infty}\frac{x(t)+\lambda(t)x(t-\tau)}{t-t_2} = -\infty \tag{25}$$

On the other hand, using condition A1 and the fact that $x(t) > 0$, $x(t-\tau) > 0$ for $t \geqq t_2$, we obtain that

$$\liminf_{t\longrightarrow\infty}\frac{1}{t-t_0}[x(t)+\lambda(t)x(t-\tau)] \geqq 0,$$

which contradicts equality (25).

This completes the proof of Theorem 3. ■

The following sufficient condition for oscillation of the solutions of problem (1), (2) is a corollary of Theorem 1 and Theorem 3.

Theorem 4. *Let conditions (H) hold as well as the conditions*

(26)
$$\liminf_{t\longrightarrow\infty}\int_{t_0}^{t}\left(1-\frac{s}{t}\right)(G(s)+\mu(s)G(s-\sigma)+F(s))\,ds = -\infty,$$

(27)
$$\limsup_{t\longrightarrow\infty}\int_{t_0}^{t}\left(1-\frac{s}{t}\right)(G(s)+\mu(s)G(s-\sigma)+F(s))\,ds = +\infty$$

for any sufficiently large number t_0, where the functions $G(t)$ and $F(t)$ are defined by (4). Then each solution $u(x,t)$ of problem (1), (2) oscillates in the domain G.

The following sufficient condition for oscillation of the solutions of problem (1), (3) is a corollary of Theorem 2 and Theorem 3.

Theorem 5. *Let conditions H1-H5 hold as well as the conditions*

$$\mu(t) \geqq 0 \text{ for } t \geqq 0 \tag{28}$$

$$\liminf_{t\longrightarrow\infty} \int_{t_0}^{t} \left(1 - \frac{s}{t}\right) F_1(s)\, ds = -\infty \tag{29}$$

$$\limsup_{t\longrightarrow\infty} \int_{t_0}^{t} \left(1 - \frac{s}{t}\right) F_1(s)\, ds = +\infty \tag{30}$$

for any sufficiently large number t_0, where the function $F_1(t)$ is defined by (19). Then each solution $u(x,t)$ of problem (1), (3) oscillates in the domain G.

Example 1. Consider the equation

$$u_{tt} + u_{tt}(x, t-\pi) - u_{xx} + u = 2e^t \cos x(\sin t + \cos t - e^{-\pi} \cdot \cos t), \tag{31}$$
$$(x,t) \in \left(0, \frac{\pi}{2}\right) \times (0,\infty) \equiv G,$$

and the boundary conditions

$$-u_x(0,t) = 0, \qquad u_x\left(\frac{\pi}{2}, t\right) = -e^t \cdot \sin t, \qquad t \geqq 0 \tag{32}$$

A straightforward verification shows that the functions

$$c(x,t,u) = u,\ f(x,t) = 2e^t \cdot \cos x \cdot (\sin t + \cos t - e^{-\pi} \cos t),$$
$$g(0,t) = 0,\ g\left(\frac{\pi}{2}, t\right) = -e^t \cdot \sin t,\ \lambda(t) \equiv 1,\ \mu(t) \equiv 0,$$
$$\gamma(x,t) \equiv 0$$

satisfy conditions (H). Moreover, from (4) we obtain that

$$G(t) = -\frac{2}{\pi} e^t \cdot \sin t, \qquad t > 0,$$
$$F(t) = \frac{4}{\pi} e^t (\sin t + \cos t - e^{-\pi} \cdot \cos t), \qquad t > 0.$$

By straightforward calculations we find that

$$I(t) = \int_{t_0}^{t} \left(1 - \frac{s}{t}\right) (G(s) + \mu(s) G(s-\sigma) + F(s))\, ds =$$
$$= e^t \cdot (t\pi)^{-1} \cdot (2\sin t - 2e^{-\pi} \sin t - \cos t) + C,$$

where C is independent of t. Hence

$$\liminf_{t\longrightarrow\infty} I(t) = -\infty, \qquad \limsup_{t\longrightarrow\infty} I(t) = +\infty,$$

i.e. conditions (26), (27) of Theorem 4 hold as well. Then from Theorem 4 it follows that each solution of problem (31), (32) oscillates in the domain $G = \left(0, \frac{\pi}{2}\right) \times (0, \infty)$. For instance, the function $u(x,t) = e^t \sin t \cos x$ is such a solution.

Example 2. Consider the equation

$$(33) \quad u_{tt} + u_{tt}(x, t-\pi) - [u_{xx} + u_{xx}(x, t-\pi)] + u = f(x,t), \qquad (x,t) \in (0,\pi) \times (0,\infty) \equiv G,$$

where $f(x,t) = e^t \cdot \sin x(2e^{-\pi} \sin t - 2\sin t + e^{-\pi} \cos t)$ and the boundary conditions

$$(34) \qquad u(0,t) = u(\pi,t) = 0, \qquad t \geqq 0.$$

It is immediately verified that the functions

$$c(x,t,u) = u, \qquad f(x,t), \qquad \lambda(t) = \mu(t) = 1$$

satisfy conditions H1-H5. Moreover, the smallest eigenvalue of the Sturm-Liouville problem

$$U'' + \alpha U = 0, \qquad U(0) = U(\pi) = 0$$

is $\alpha_0 = 1$ and the corresponding eigenfunction is $\varphi(x) = \sqrt{\frac{2}{\pi}} \sin x > 0$, $x \in (0,\pi)$. Then from (19) we find that

$$F_1(t) = \int_0^\pi f(x,t)\sqrt{\frac{2}{\pi}} \sin x\, dx \cdot \left(\int_0^\pi \sqrt{\frac{2}{\pi}} \sin x\, dx\right)^{-1} = \frac{\pi}{4} e^t (2e^{-\pi} \sin t - 2\sin t + e^{-\pi}\cos t).$$

By straightforward calculations we obtain that

$$I_1(t) = \int_{t_0}^t \left(1 - \frac{s}{t}\right) F_1(s)\, ds = \frac{\pi}{4} \cdot \frac{e^t}{t}\left(\cos t - e^{-\pi}\cos t + \frac{1}{2} e^{-\pi} \sin t\right) + C,$$

where C is independent of t. Hence $\liminf_{t\longrightarrow\infty} I_1(t) = -\infty$ and $\limsup_{t\longrightarrow\infty} I_1(t) = +\infty$, i.e. conditions (29), (30) of Theorem 5 hold as well. Then from

Theorem 5 it follows that each solution of problem (33), (34) oscillates in the domain $G = (0, \pi) \times (0, \infty)$. For instance, the function $u(x,t) = e^t \sin x \cdot \cos t$ is such a solution.

In the subsequent theorems we shall restrict our attention to some particular cases of equation (1) for which new sufficient conditions for oscillation of the solutions are obtained.

Case A.

Assume that $\lambda(t) \equiv 0$. We shall use the following result of T. Kusano and M. Naito [5] concerning differential inequalities of the form

$$(q(t)(p(t) \cdot x)')' + h(t,x) \leqq r(t), \qquad t \geqq t_0 \tag{35}$$

We shall say that conditions (B) are satisfied if the following conditions hold:

B1. $p(t),\ q(t) \in C([t_0, \infty);\ (0, \infty))$, $\int_{t_0}^{\infty} (q(t))^{-1}\, dt = \infty$.

B2. $h(t,x) \in C([t_0, \infty) \times (0, \infty);\ (0, \infty))$, $h(t,x)$ is a monotone increasing function of its second argument x.

B3. $r(t) \in C([t_0, \infty);\ \mathbb{R})$.

Theorem 6 [5]. *Let conditions (B) hold and let the differential inequality*

$$(q(t)(p(t)x)')' + h(t,x) \leqq 0 \tag{36}$$

have no eventually positive solutions. Moreover, let a function $\theta(t) \in C^2([t_0, \infty);\ \mathbb{R})$ exist with the following properties:

(37) *$\theta(t)$ takes both positive and negative values for arbitrarily large values of t.*

$$(q(t)(p(t) \cdot \theta(t))')' = r(t), \qquad t \geqq t_0 \tag{38}$$

$$\liminf_{t \longrightarrow \infty} [p(t) \cdot \theta(t)] = 0 \tag{39}$$

Then the differential inequality (35) has no eventually positive solutions.

The following sufficient condition for oscillation of the solutions of problem (1), (2) in the case when $\lambda(t) \equiv 0$ is a corollary of Theorem 1 and Theorem 6.

Theorem 7. *Let the following conditions be fulfilled:*

1. *Conditions (H) hold.*
2. *The function $h(u)$ is monotone increasing in the interval $(0,\infty)$.*
3. *The differential inequality*

$$x''(t) + p(t)\cdot h(x(t)) \leqq 0, \qquad t \geqq t_0$$

has no eventually positive solutions.

4. *There exists a function $\theta(t) \in C^2([t_0,\infty);\mathbb{R})$ with the following properties:*
   a) *$\theta(t)$ takes both positive and negative values for arbitrarily large values of t.*
   b) $[\theta(t)]'' = G(t) + \mu(t)G(t-\sigma) + F(t), \qquad t \geqq t_1$
   c) $\lim_{t\to\infty} \theta(t) = 0.$

Then each solution $u(x,t)$ of problem (1), (2) oscillates in the domain G.

Example 3. Consider the equation

$$u_{tt}(x,t) - u_{xx}(x,t) - u_{xx}(x,t-\pi) + 2u = f(x,t), \qquad (x,t) \in \left(0,\frac{\pi}{2}\right) \times (0,\infty), \tag{40}$$

where $f(x,t) = e^{-t}\cdot\cos x\cdot(3\sin t - 2\cos t - e^{\pi}\sin t)$ and the boundary condition

$$-u_x(0,t) = 0,\ u_x\left(\frac{\pi}{2},t\right) = -e^{-t}\sin t, \qquad t > 0. \tag{41}$$

It is immediately verified that the functions

$$c(x,t,u) = 2u, \qquad f(x,t), \qquad \lambda(t) \equiv 0, \qquad \mu(t) \equiv 1,$$
$$\gamma(x,t) \equiv 0, \qquad g(0,t) = 0, \qquad g\left(\frac{\pi}{2},t\right) = -e^{-t}\cdot\sin t$$

satisfy conditions (H). Moreover, from (4) we obtain

$$G(t) = -\frac{2}{\pi}e^{-t}\sin t, \qquad t > 0$$
$$F(t) = \frac{2}{\pi}e^{-t}\cdot(3\sin t - 2\cos t - e^{\pi}\sin t), \qquad t > 0.$$

Then

$$I_2(t) = \int_{t_0}^{t}\left(1-\frac{s}{t}\right)(G(s)+\mu(s)G(s-\sigma)+F(s))\,ds =$$
$$= \int_{t_0}^{t}\left(1-\frac{s}{t}\right)\cdot\frac{2}{\pi}e^{-s}\cdot(3\sin s - 2\cos s - e^{\pi}\sin s)\,ds,$$

which immediately implies that $\lim_{t \to \infty} I_2(t) < \infty$. Hence conditions (26) and (27) of Theorem 4 are not satisfied. It is easy to check that the differential inequality $x'' + 2x \leqq 0$ has no eventually positive solutions. Let $\theta(t) = \frac{e^{-t}}{\pi}[2\sin t + (3 - e^{\pi})\cos t]$. Then for $n \in \mathbb{Z}$ we obtain

$$\theta\left(\frac{\pi}{2} + 2n\pi\right) > 0, \qquad \theta\left(\frac{3\pi}{2} + 2n\pi\right) < 0.$$

Moreover, $[\theta(t)]'' = \frac{2}{\pi}e^{-t} \cdot (3\sin t - 2\cos t - e^{\pi}\sin t) = G(t) + G(t - \pi) + F(t)$ and $\lim_{t \to \infty} \theta(t) = 0$. Hence the function $\theta(t)$ satisfies condition 4 of Theorem 7. Then by Theorem 7 each solution $u(x, t)$ of problem (40), (41) oscillates in the domain $G = \left(0, \frac{\pi}{2}\right) \times (0, \infty)$. For instance, the function $u(x, t) = e^{-t}\sin t \cos x$ is such a solution.

We shall note that a result analogous to that of Theorem 7 can be obtained for problem (1), (3) as well.

Case B.

Assume that $f(x, t) \equiv 0$, $g(x, t) \equiv 0$. In this case the finding of sufficient conditions for oscillation of the solutions of equation (1) in the domain G is reduced to the investigation of the oscillatory properties of differential inequalities of neutral type of the form

$$(42) \quad \frac{d^2}{dt^2}[x(t) + \lambda(t)x(t-\tau)] + q_0(t)x(t) + q(t)x(t-\sigma) + p(t) \cdot h(x(t)) \leqq 0, \qquad t \geqq t_0,$$

$$(43) \quad \frac{d^2}{dt^2}[x(t) + \lambda(t)x(t-\tau)] + q_0(t)x(t) + q(t)x(t-\sigma) + p(t) \cdot h(x(t)) \geqq 0, \qquad t \geqq t_0.$$

Together with (42) and (43) we shall consider the nonlinear differential equation of neutral type

$$(44) \quad \frac{d^2}{dt^2}[x(t) + \lambda(t)x(t-\tau)] + q_0(t)x(t) + q(t)x(t-\sigma) + p(t) \cdot h(x(t)) = 0, \qquad t \geqq t_0.$$

We shall say that conditions (C) are satisfied if the following conditions hold:

C1. $\lambda(t) \in C^2([t_0, \infty); \mathbb{R})$,
$0 < \lambda_1 \leqq \lambda(t) \leqq \lambda_2$ for $t \geqq t_0$, λ_1, $\lambda_2 = \text{const}$.
C2. $q_0(t)$, $q(t) \in C([t_0, \infty); [0, \infty))$,
C3. $p(t) \in C([t_0, \infty); (0; \infty))$,
C4. $h(u) \in C(\mathbb{R}; \mathbb{R})$, $h(-u) = -h(u)$,

$h(u)$ is a positive and monotone increasing function in the interval $(0, \infty)$.

Theorem 8. *Let the following conditions be satisfied:*

1. *Conditions (C) hold.*
2. *For any closed and measurable set $E \subset [t_0, \infty)$ for which $meas(E \cap [t, t+2\tau]) \geqq \tau$, $t \in [t_0, \infty)$, the following condition holds*

$$\int_E p(t)\,dt = \infty \tag{45}$$

Then:

(i) *the differential inequality (42) has no eventually positive solutions;*
(ii) *the differential inequality (43) has no eventually negative solutions;*
(iii) *all solutions of the differential equation (44) oscillate.*

Proof:

(i) Let $x(t)$ be an eventually positive solution of the differential inequality (42). Then there exists a number $t_1 \geqq t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma) > 0$ for $t \geqq t_1$. From conditions C2-C4 and (42) it follows that

$$\begin{aligned} \frac{d^2}{dt^2}[x(t) + \lambda(t)x(t-\tau)] \leqq -q_0(t)x(t) - q(t)x(t-\sigma) - \\ - p(t) \cdot h(x(t)) \leqq -p(t) \cdot h(x(t)) < 0, \qquad t \geqq t_1. \end{aligned} \tag{46}$$

Hence the function $\frac{d}{dt}[x(t) + \lambda(t)x(t-\tau)]$ is monotone decreasing in the interval $[t_1, \infty)$. Suppose that there exists a number $t_2 \geqq t_1$ such that $\frac{d}{dt}[x(t_2) + \lambda(t_2)x(t_2 - \tau)] = -c < 0$. Then for any point $t \geqq t_2$ the following inequality holds

$$\frac{d}{dt}[x(t) + \lambda(t)x(t-\tau)] \leqq \frac{d}{dt}[x(t_2) + \lambda(t_2)x(t_2-\tau)] = -c$$

Integrate last inequality over the interval $[t_2, t]$, $t > t_2$ and obtain

$$x(t) + \lambda(t)x(t-\tau) \leqq x(t_2) + \lambda(t_2)x(t_2-\tau) - c(t-t_2).$$

Hence $\limsup\limits_{t \longrightarrow \infty}[x(t) + \lambda(t)x(t-\tau)] \leqq 0$ which contradicts the assumption that $x(t)$ is an eventually positive solution. Hence

$$\frac{d}{dt}[x(t) + \lambda(t) \cdot x(t-\tau)] \geqq 0, \qquad t \geqq t_1 \tag{47}$$

whence we obtain that $x(t) + \lambda(t)x(t-\tau) \geqq c_1 > 0$ for $t \geqq t_1$. From Lemma 1 [9], [11] it follows that there exists a closed and measurable set $E \subset [t_1, \infty)$ and a constant $c_2 > 0$ such that $x(t) \geqq c_2$ for $t \in E$ and $meas(E \cap [t, t+2\tau]) \geq \tau$ for $t \geqq t_1$. Then from condition C4 it follows that

$$h(x(t)) \geqq h(c_2) = c_3 > 0 \text{ for } t \in E.$$

Integrate both sides of inequality (46) over the interval $[t_1, t]$, $t > t_1$ and using (47), we obtain

$$C_3 \int_{E\cap[t_1,t]} p(s)\,ds \leqq \int_{t_1}^{t} p(s)h(x(s))\,ds \leqq \frac{d}{dt}[x(t_1) + \lambda(t_1)x(t_1-\tau)] -$$
$$- \frac{d}{dt}[x(t) + \lambda(t)x(t-\tau)] \leqq \frac{d}{dt}[x(t_1) + \lambda(t_1)x(t_1-\tau)] = C_4.$$

For $t \longrightarrow \infty$ from the above inequality it follows that $\int_E p(t)\,dt < \infty$, which contradicts condition (45). Thus assertion (i) of Theorem 8 is proved.

(ii) The proof follows immediately from the fact that if $x(t)$ is an eventually negative solution of the differential inequality (43), then $-x(t)$ is an eventually positive solution of the differential inequality (42).

(iii) The proof follows immediately from assertions (i) and (ii). ∎

The following sufficient condition for oscillation of the solutions of problem (1), (2) or (1), (3) in the case when $f(x,t) \equiv 0$ and $g(x,t) \equiv 0$ is a corollary of Theorem 1, Theorem 2 and Theorem 8.

Theorem 9. *Let the following conditions hold:*

1. *Conditions (H) are fulfilled.*
2. $0 < \lambda_1 \leqq \lambda(t) < \lambda_2$, $t \geqq t_0$, λ_1, $\lambda_2 = \text{const}$.
3. $h(-u) = -h(u)$, $u \in \mathbb{R}$; $h(u)$ *is a monotone increasing function in the interval* $(0, \infty)$.
4. *For any closed and measurable set* $E \subset [t_0, \infty)$ *for which* $meas(E \cap [t, t+2\tau]) \geqq \tau$, $t \in [t_0, \infty)$ *the following condition holds*

$$\int_E p(t)\,dt = \infty \tag{48}$$

Then each solution $u(x,t)$ of problem (1), (2) or (1), (3) oscillates in the domain G.

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